Using polynomials with set valued coefficients to solve relaxed constraint satisfaction problems

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Outline of the presentation

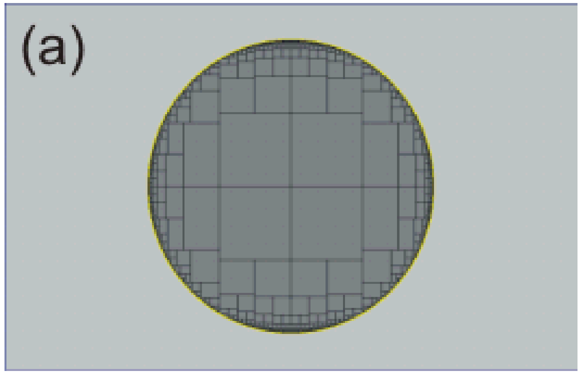
- **Relaxed** Constraint Satisfaction Problems (**CSP**)
- Solution of the relaxed CSP assuming a **fixed amount** of constraints **to be relaxed**
- **Demo**
- New representation of the solution of a relaxed CSP in the form of **set valued polynomials**

Solving a CSP

Consider the following Constraint Satisfaction Problem (CSP)

$$\begin{cases} C_1 : & f_1(\mathbf{x}) = 0 \\ \dots & \dots \\ C_n : & f_n(\mathbf{x}) = 0 \end{cases}$$

$$\mathbf{x} \in \mathbb{R}^m$$



$C_1 : x^2 + y^2 \in [0, r^2]$

Compute the solution set for each constraint

$$\mathbb{X}_i = \{\mathbf{x} \in \mathbb{R}^m, \mathbf{x} \text{ satisfies } C_i \textit{ i.e. } f_i(\mathbf{x}) = 0\}$$

$$= f_i^{-1}(\mathbf{0}).$$

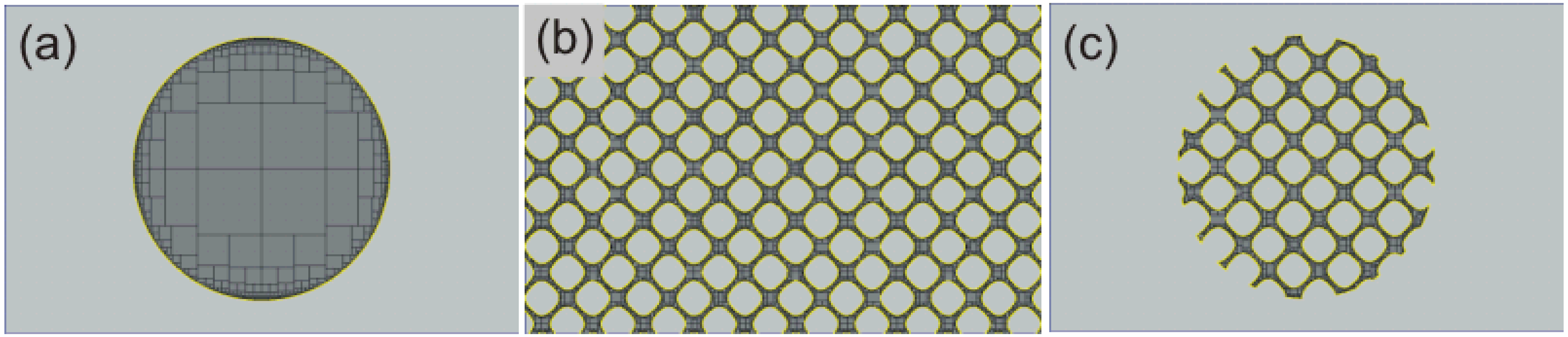
Solving is intersecting

$$\begin{cases} C_1 : & f_1(\mathbf{x}) = 0 \\ \ldots & \ldots \\ C_n : & f_n(\mathbf{x}) = 0 \end{cases}$$

$$\mathbf{x} \in \mathbb{R}^m$$

$$\mathbb{X}_i = f_i^{-1}(\mathbf{0}).$$

$$\mathbb{S} = \bigcap_{i \in \{0..n\}} \mathbb{X}_i$$

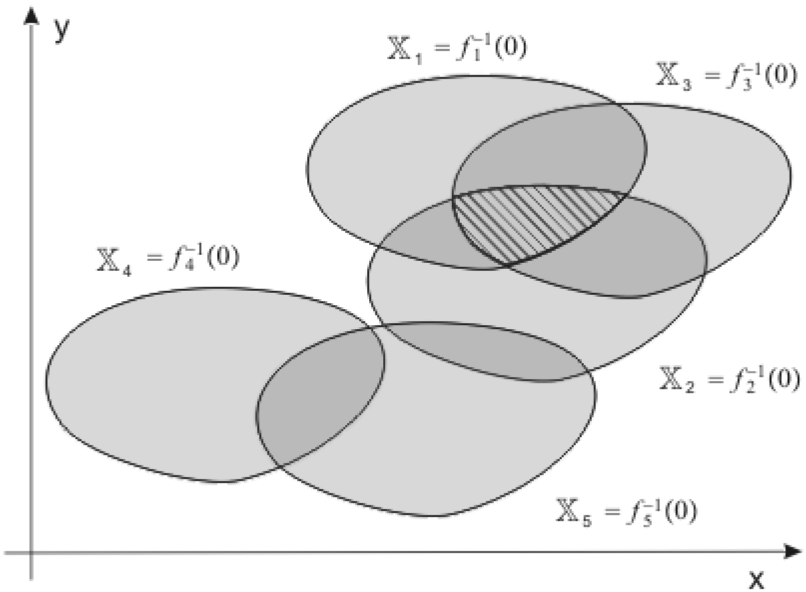


$C_1 : x^2 + y^2 \in [0, r^2]$ $C_2 : \sin 4x - \cos 4y \in [-\varepsilon, \varepsilon]$ $C_1 \& C_2$

Relaxed intersection

$$\begin{cases} f_1(x,y) = 0 \\ f_2(x,y) = 0 \\ f_3(x,y) = 0 \\ f_4(x,y) = 0 \\ f_5(x,y) = 0 \end{cases}$$

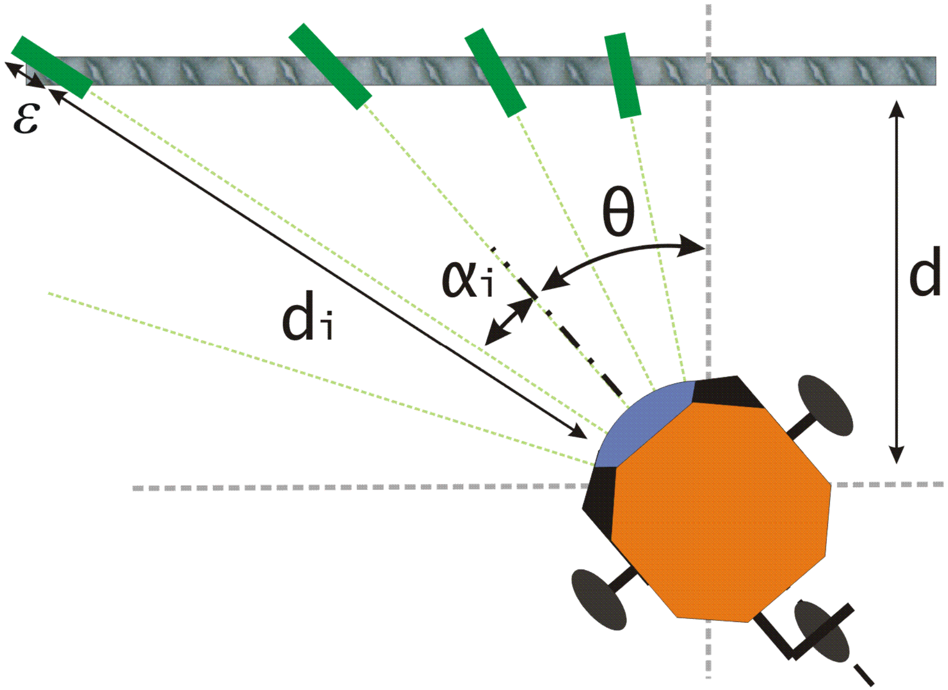
$$(x,y) \in \mathbb{R}^2$$



relaxed intersection illustration on $\mathbb{R}^2$ sets

$$\mathbb{S}_q = \bigcap_{i \in \{0..n\}}^{\{q\}} \mathbb{X}_i = \{\mathbf{x} \in \mathbb{R}^m, \exists \mathbb{I} \subset \{1,..,n\}, \quad card(\mathbb{I}) = n - q, \forall i \in \mathbb{I}, \mathbf{x} \in \mathbb{X}_i\}$$

Example of localization

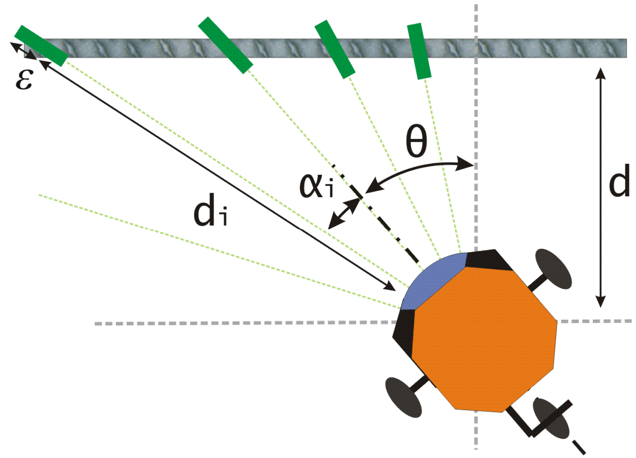


Example

Setting up the equations

For each measurement « Constraint »

$$d = d_i \cos(\theta + \alpha_i)$$



System of equations (CSP)

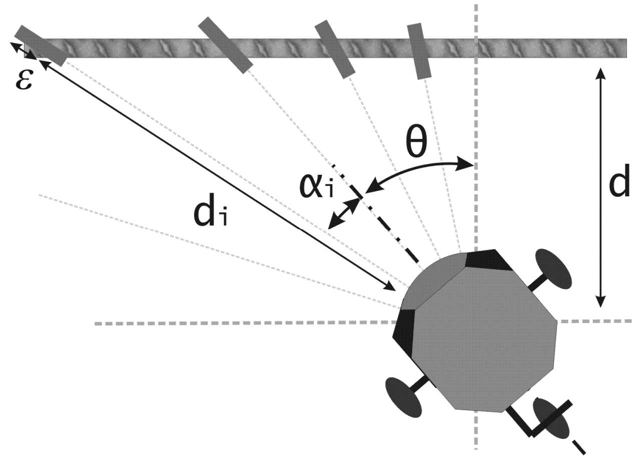
$$\begin{cases} d - d_1 \cos(\theta + \alpha_1) = 0 \\ d - d_2 \cos(\theta + \alpha_2) = 0 \\ \dots \\ d - d_m \cos(\theta + \alpha_m) = 0 \end{cases}$$

$$d \in [0, \infty], \\ \theta \in [-\tfrac{\pi}{2}, \tfrac{\pi}{2}].$$

Software demonstration

Setting up the equations

>>DEMO<<



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$d \in [0, \infty]$,
$\theta \in [-\frac{\pi}{2}, \frac{\pi}{2}]$.

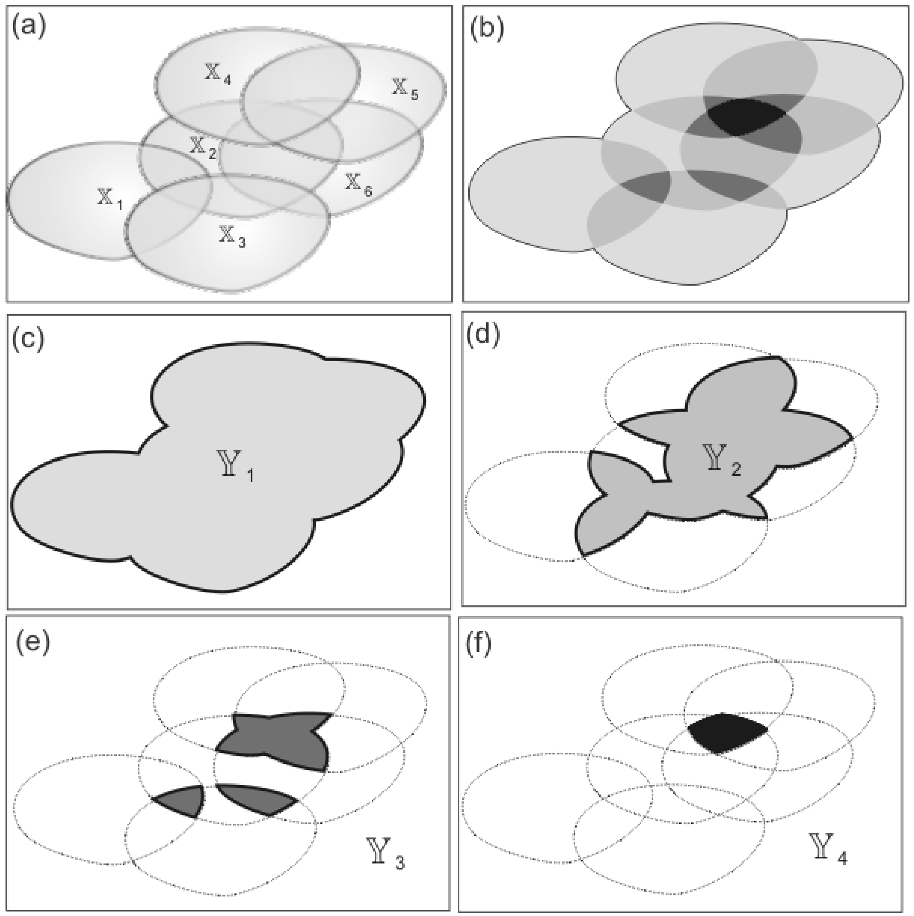
New representation

Consider the following transform

$$\mathcal{T}: (\mathcal{P}(\mathbb{R}^2))^n \rightarrow (\mathcal{P}(\mathbb{R}^2))^n$$
$$(\mathbb{X}_1,..,\mathbb{X}_n) \mapsto (\mathbb{Y}_1,..,\mathbb{Y}_n)$$

Such as

$$\mathbb{Y}_k = \bigcap_{i\in\{1,..,n)}^{\{n-k\}} \mathbb{X}_i$$



Some properties of the transform

Consider the following transform

$$\mathcal{T}: (\mathcal{P}(\mathbb{R}^2))^n \rightarrow (\mathcal{P}(\mathbb{R}^2))^n$$
$$(\mathbb{X}_1, .., \mathbb{X}_n) \mapsto (\mathbb{Y}_1, .., \mathbb{Y}_n)$$

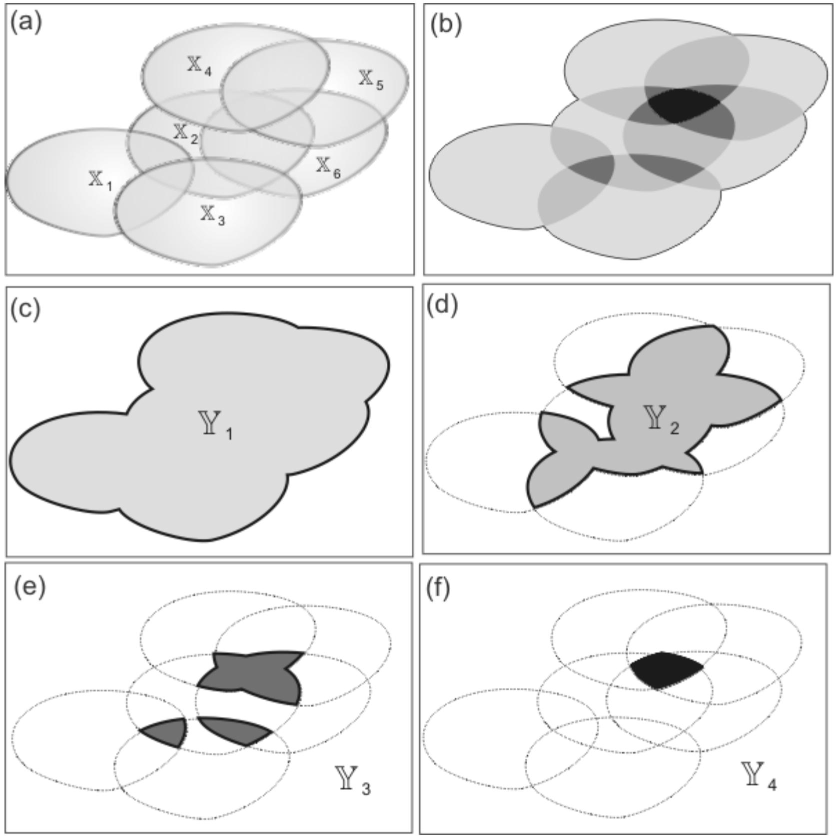
The vector $(\mathbb{Y}_1, .., \mathbb{Y}_n)$ is **sorted**

i.e. $\forall k, \mathbb{Y}_{k+1} \subset \mathbb{Y}_k$

The transform is **idempotent**

i.e. $\mathcal{T} \circ \mathcal{T} = \mathcal{T}$

$\mathcal{T}$ is the « sort transform »



Polynomials with set valued coefficients

Polynomial with set valued coefficients

$$X(s) = \sum_{i=0}^{n} \mathbb{X}_i s^i$$

example

$$X(s) = [-2,3]s^3 + [1,\infty]s^2 + [2,4]s + [5,8].$$

Operations on polynomials (+,*)

$$\begin{aligned}(\mathbb{A}_1 s + \mathbb{A}_0) * (\mathbb{B}_1 s + \mathbb{B}_0) &= (\mathbb{A}_1 * \mathbb{B}_1)s^2 + (\mathbb{A}_1 * \mathbb{B}_0 + \mathbb{A}_0 * \mathbb{B}_1)s + \mathbb{A}_0 * \mathbb{B}_0 \\ &= (\mathbb{A}_1 \cap \mathbb{B}_1)s^2 + (\mathbb{A}_1 \cap \mathbb{B}_0) \cup (\mathbb{A}_0 \cap \mathbb{B}_1)s + \mathbb{A}_0 \cap \mathbb{B}_0.\end{aligned}$$

$$\begin{aligned}(\mathbb{A}_1 s + \mathbb{A}_0) + (\mathbb{B}_1 s + \mathbb{B}_0) &= (\mathbb{A}_1 + \mathbb{B}_1)s + (\mathbb{A}_0 + \mathbb{B}_0) \\ &= (\mathbb{A}_1 \cup \mathbb{B}_1)s + \mathbb{A}_0 \cap \mathbb{B}_0.\end{aligned}$$

Polynomial notation of the sort transform

Denote by

$$X^*(s) = \prod_{i=1}^{n} (\mathbb{X}_i s + \top) \quad \text{where} \quad \top = \mathbb{R}^m$$

Denote by

$$X^*(s) = \sum_{i=1}^{n} \mathbb{X}_i^* s^i + \top$$

We have

$$(\mathbb{X}_1^*, .., \mathbb{X}_n^*) = \mathcal{T}(\mathbb{X}_1, .., \mathbb{X}_n)$$

$X^*(s)$ is the «polynomial sort transform »

Solving the CSP

relaxed CSP

$$CSP\{\mathbf{x} \in \mathbb{R}^m, \{C_1, \ldots, C_n\}\}$$

variables

domain

constraints

solution sets

$$\forall i, \mathbb{X}_i = \{\mathbf{x} \in \mathbb{R}^m, \mathbf{x} \text{ satisfies } C_i\}$$

polynomial sort transform

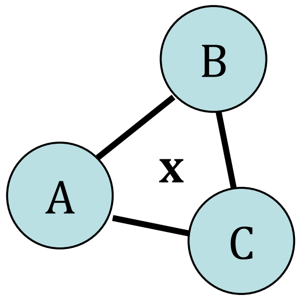
$$X^*(s) = \prod_{i=1}^{n} (\mathbb{X}_i s + \top)$$

Solving distributed CSP

Process A : $CSP\_A\{\mathbf{x} \in \mathbb{R}^m, \{C_1, \ldots, C_r\}\} \Rightarrow X_A^*(s) = \prod_{i=1}^{r} (\mathbb{X}_i s + \top)$

Process B : $CSP\_B\{\mathbf{x} \in \mathbb{R}^m, \{C_{r+1}, \ldots, C_p\}\} \Rightarrow X_B^*(s) = \prod_{i=r+1}^{p} (\mathbb{X}_i s + \top)$

Process C : $CSP\_C\{\mathbf{x} \in \mathbb{R}^m, \{C_{p+1}, \ldots, C_n\}\} \Rightarrow X_C^*(s) = \prod_{i=p+1}^{n} (\mathbb{X}_i s + \top)$



The system : $CSP\{\mathbf{x} \in \mathbb{R}^m, \{C_1, \ldots, C_n\}\} \Rightarrow X^*(s) = \prod_{i=1}^{n} (\mathbb{X}_i s + \top)$

$$X^*(s) = X_A^*(s) * X_B^*(s) * X_C^*(s)$$

Algorithm : Contractor polynomials

Contractor polynomial sort transform

$$\mathcal{C}^*(s) = \prod_{i=1}^{n} (\mathcal{C}_i s + \mathcal{C}^\top) \quad / \quad \mathcal{C}^*(s) = \sum_{i=1}^{n} \mathcal{C}_i^* s^i + \mathcal{C}^\top$$

Where

$$\forall [\mathbf{x}] \in \mathbb{IR}^m, \mathcal{C}^\top([\mathbf{x}]) = [\mathbf{x}]$$

$$\forall (i,j), \mathcal{C}_i + \mathcal{C}_j = \mathcal{C}_i \sqcup \mathcal{C}_j$$

$$\forall (i,j), \mathcal{C}_i * \mathcal{C}_j = \mathcal{C}_i \sqcap \mathcal{C}_j$$

Note that $\mathcal{C}^*(s)([\mathbf{x}])$ is a polynomial with box valued coefficients

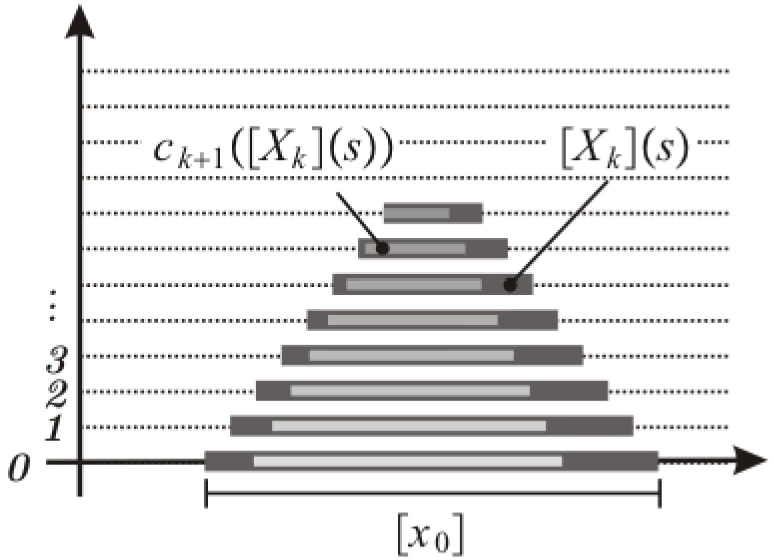
This polynomial theory can be extended to cases where coefficients are defined in a set having **lattice structure**

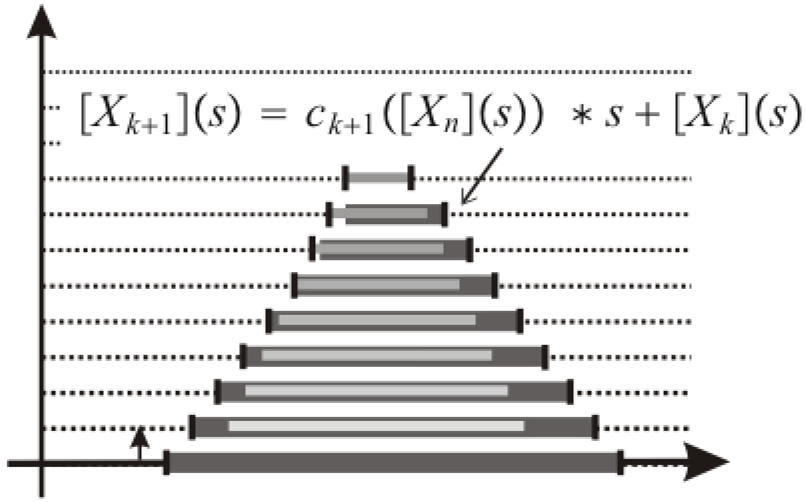
Demo

>>DEMO<<

- Finding the intersection of lines

$$\mathcal{C}^*(s) = \prod_{i=1}^{n}(\mathcal{C}_i s + \mathcal{C}^\top) \longrightarrow \mathcal{C}^*(s) = (\mathcal{C}_n s + \mathcal{C}^\top) \circ \ldots \circ (\mathcal{C}_1 s + \mathcal{C}^\top).$$

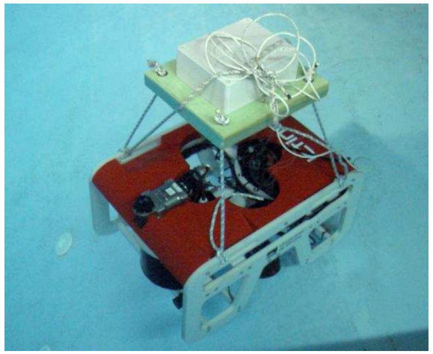


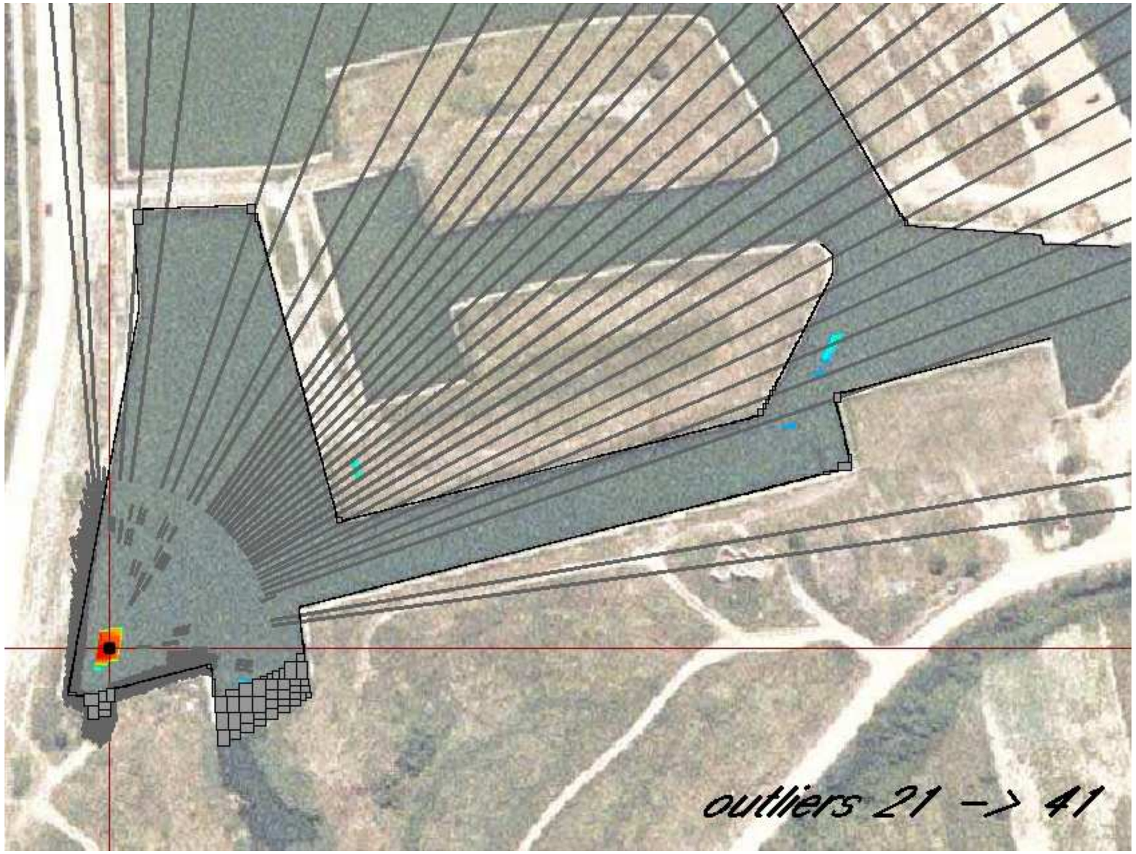


Real application

➢ Using Girona University dataset to compute the trajectory their robot made

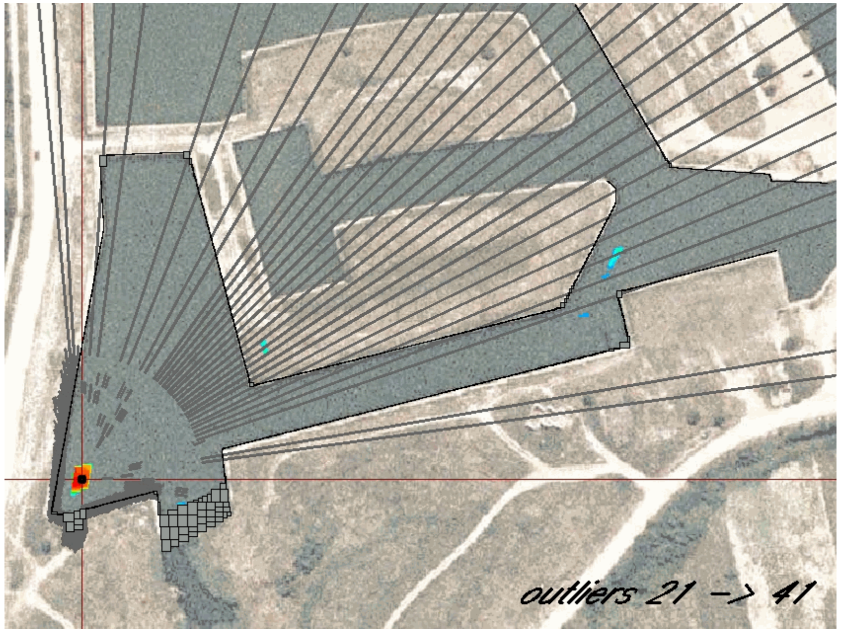
Girona University
Ictineu AUV





Results

Dynamic localization



Global localization